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A complete characterization of phase space measurements

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Abstract
We characterize all the phase space measurements for a non-relativistic particle.

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1. Introduction

In the usual framework of quantum mechanics, the states (density matrices) of a physical system are described by positive trace class trace one operators acting on a Hilbert space $\mathcal{H}$, and the physical quantities (observables) are associated with self-adjoint operators on $\mathcal{H}$ in such a way that $\mathrm{Tr}(SA)$ is the expectation value of the observable A when the system is in the state S (here Tr denotes the trace).

Nevertheless, a careful analysis of measurement processes shows that one has to generalize suitably the concept of observable for both theoretical and experimental reasons [1–4]. These *generalized observables* are described as mathematical objects by *positive operator valued measures* (POVM). In this framework, one can describe measurements of quantities such as angle of rotation, phase and arrival times, as well as joint measurements of quantities such as position and momentum, incompatible according to the standard textbook formulation of quantum mechanics.

In order to give a physical meaning to the observables one invokes some properties of covariance with respect to a symmetry group. The requirement of covariance is a strong constraint: it allows us to select the measurements of physical interest among the larger class of all the possible generalized observables. As recently proved, from this principle follows not only the characterization of generalized observables, but also the determination of generators of quantum dynamical semigroups [5–7].

In this paper, we classify all the possible joint observables of position and momentum that arise from the request of covariance with respect to the Galilei group. In the literature these observables are usually called *phase space measurements* for a non-relativistic particle. We restrict our attention to the isochronous Galilei group since the POVMs covariant with respect to this group have a clear and transparent physical meaning. Moreover the technical treatment (compare proposition 4) is rather simple. We have in mind the possibility of treating more general spacetime groups (e.g., Poincaré, de Sitter).

The quest for the characterization of phase space measurements in quantum mechanics goes back to the 1970s, and in particular to the seminal works of Ali and Prugovečki [8] and Holevo [2, 9]; the first is concerned with the representation of quantum mechanics on fuzzy phase space and the second with a general treatment of quantum measurements covariant with respect to a given symmetry group. The result presented in this paper, which relies on a previous work on the characterization of POVM covariant with respect to an irreducible representation of a symmetry group [10], essentially confirms the previous ones showing, along a different line of proof, that indeed all phase space measurements for a non-relativistic particle are expressed in terms of an operator-valued density, thus releasing the more restrictive assumptions considered in [8] (see also [3]) and putting into evidence with respect to [2, 9] that square-integrability of the considered representation is both a sufficient and necessary condition.

The paper is organized in the following way. In section 2, we briefly review the physical motivations that justify the introduction of covariant positive operator valued measures from the point of view of quantum measurement theory. In section 3, we give the complete classification of the phase space measurements for a non-relativistic particle. The proof of the result is given in section 4.

2. A brief review of POVMs

For an exhaustive exposition of the theory of covariant POVMs from the perspective of quantum measurement theory, one can refer to [2–4]. However, for the reader's convenience, we briefly recall the main steps which lead quite naturally to the idea of covariant POVM.

First of all, we recall the mathematical definition of POVM.

Definition 1. *Let X be a metric space and $\mathcal{H}$ a (complex separable) Hilbert space. A map E from the Borel subsets $\mathcal{B}(X)$ of X into the set $\mathcal{L}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ such that*

(i) $\langle \phi, E(Z)\phi\rangle \geqslant 0 \;\forall\, \phi \in \mathcal{H}, Z \in \mathcal{B}(X)$

(ii) $E(X) = I$

(iii) $E(\cup_i Z_i) = \sum_i E(Z_i)$ *for all disjoint sequences of subsets (the series converging in the weak sense),*

is called a (normalized) positive operator valued measure (POVM) based on X.

The role of POVMs in quantum mechanics is justified by the following observation. Given a physical quantity described by a self-adjoint operator A, it is well known how one obtains the probability distribution of the outcomes of A. Indeed by the spectral theorem, A uniquely defines a *projection-valued measure* P, i.e. a map

$$P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \tag{1}$$

from the Borel subsets $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ into the space of bounded operators $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$ satisfying the following three conditions:

(i) $P(Z)$ is an orthogonal projection operator for all $Z \in \mathcal{B}(\mathbb{R})$, that is,

$$P(Z) = P^*(Z) = P(Z)^2 \qquad \forall\, Z \in \mathcal{B}(\mathbb{R}) \tag{2}$$

(ii) $P(\mathbb{R}) = I$

(iii) $P(\cup_i Z_i) = \sum_i P(Z_i)$ for all disjoint sequences of subsets (the series converging in the weak sense).

Comparing with definition 1, one easily checks that a projection-valued measure is a particular case of POVM. With this notation, the physical content of quantum theory is based on the following assumption: if one measures the observable A when the system is in a state S, the probability to have an outcome in Z is given by $\mathrm{Tr}[SP(Z)]$.

The fact that P is a projection-valued measure assures that the map

$$Z \mapsto \mathrm{Tr}[SP(Z)] =: \mu_S^A(Z) \tag{3}$$

is a probability distribution on $\mathbb{R}$. Clearly, the physical content of the observable A is completely given by the map

$$S \mapsto \mu_S^A$$

from the set of states into the space of probability measure on $\mathbb{R}$ (the above map is usually called a *measurement*).

The key remark is that in order that equation (3) defines a probability measure, it is sufficient and necessary to replace equation (2) with the weaker condition that $P(Z)$ is a positive operator, that is

$$\langle \phi, P(Z)\phi \rangle \geqslant 0 \qquad \forall\, \phi \in \mathcal{H} \quad Z \in \mathcal{B}(\mathbb{R}). \tag{4}$$

Then the corresponding map $Z \mapsto P(Z)$ will be a positive operator valued measure on $\mathbb{R}$.

Moreover, in order to take into account joint measurements, another generalization suggested by this approach consists in assuming that the space of measurement outcomes is an arbitrary metric space X instead of $\mathbb{R}$. For example, the joint measurements of position along the three axes of the Euclidean space defines a projection measure on $X = \mathbb{R}^3$.

Given a POVM E on the space X, by the above discussion, it is reasonable to define a *generalized* measurement associated with E as a map from the set of states to the space of probability measures on X

$$S \mapsto \mu_S^E$$

with μ_S^E defined according to equation (3)

$$Z \mapsto \mathrm{Tr}[SE(Z)] = \mu_S^E(Z).$$

This mathematical framework can be further enriched by introducing the concept of POVM *covariant* with respect to a symmetry group. From a mathematical point of view, one has the following definition.

Definition 2. *Let G be a group that acts both on $\mathcal{H}$ by means of a projective unitary representation U and on the outcome space X by a geometrical (left) action α. A POVM E on X is said to be* covariant *with respect to G if, for all $g \in G$,*

$$U_g E(Z) U_g^* = E(\alpha_g(Z)) \qquad \forall\, Z \in \mathcal{B}(X). \tag{5}$$

In order to explain the physical meaning of equation (5), let us fix the ideas on a simple example and give a natural definition of a position measurement on the real line $\mathbb{R}$, on which $\mathbb{R}$ itself acts as the group of translations. If $x \in \mathbb{R}$, its action on an element $y \in \mathbb{R}$ is $\alpha_x(y) = x+y$. If S is a state, denote with xS the translate of S by x. In order that a measurement E be a position measurement, the probability distribution of the outcomes of E performed on S and xS should satisfy the following relation:

$$\mu^E_{xS}(Z+x) = \mu^E_S(Z) \qquad \forall\, Z \in \mathcal{B}(\mathbb{R}) \quad x \in \mathbb{R}. \tag{6}$$

In the more general setting in which a generic transformation group G acts both on the quantum system and on the outcome space X, the above condition reads

$$\mu^E_{gS}(\alpha_g(Z)) = \mu^E_S(Z) \qquad \forall\, Z \in \mathcal{B}(X),\, g \in G. \tag{7}$$

Since the action of $g \in G$ on the state S is given by

$$gS = U_g S U_g^*$$

a straightforward calculation shows that equations (3) and (7) imply the covariance condition (5).

In particular, if X is the (classical) phase space of the system on which the isochronous Galilei group acts, the POVMs based on X and satisfying equation (5) are called *phase space measurements*.

3. Phase space measurements

In the present section, we characterize all the phase space measurements of a non-relativistic particle of mass m. For the sake of simplicity we restrict to the spinless case, the extension to the general case being straightforward.

Every observer describes the phase space associated with a free particle as $X = \mathbb{R}^3 \times \mathbb{P}^3$. The symmetry group is the isochronous Galilei group $G = (\mathbb{R}^3 \times \mathbb{V}^3) \times' SO(3)$, where $\mathbb{R}^3$ is the three-dimensional vector group of space translations, $\mathbb{V}^3$ is the three-dimensional vector group of Galilean boosts and $SO(3)$ is the group of rotations (connected with the identity). In particular, the composition law of G is given by

$$(\vec{a}, \vec{v}, R)(\vec{a}', \vec{v}', R') = (\vec{a} + R\vec{a}', \vec{v} + R\vec{v}', RR').$$

The action of an element $g = (\vec{a}, \vec{v}, R) \in G$ on a point $(\vec{q}, \vec{p}) \in X$ is given by

$$\alpha_g(\vec{q}, \vec{p}) = (\vec{a} + R\vec{q}, m\vec{v} + R\vec{p}). \tag{8}$$

The Hilbert space of a non-relativistic spinless particle of mass m is $\mathcal{H} = L^2(\mathbb{R}^3, \mathrm{d}\vec{x})$, and G acts on $\mathcal{H}$ by means of the irreducible projective unitary representation U given by

$$[U_{(\vec{a},\vec{v},R)}\phi](\vec{x}) = \mathrm{e}^{\mathrm{i}m\vec{v}\cdot(\vec{x}-\vec{a})}\phi(R^{-1}(\vec{x}-\vec{a})). \tag{9}$$

With this notation, the problem of determining the phase space measurements reduces to the characterization of the POVM on X covariant with respect to U. The following theorem faces up this problem.

Theorem 3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a positive trace class trace one operator such that*

$$TU_{(\vec{0},\vec{0},R)} = U_{(\vec{0},\vec{0},R)}T \qquad \forall\, R \in SO(3) \tag{10}$$

i.e. T is a density matrix invariant under rotations. For all $Z \in \mathcal{B}(X)$, let $E_T(Z)$ be the operator

$$E_T(Z) = \frac{1}{(2\pi)^3}\int_Z U_{(\vec{a},\frac{\vec{p}}{m},I)} T U^*_{(\vec{a},\frac{\vec{p}}{m},I)}\, \mathrm{d}\vec{a}\,\mathrm{d}\vec{p} \tag{11}$$

where the integral is understood in the weak sense.

The map $Z \mapsto E_T(Z)$ is a POVM on X covariant with respect to U.

Conversely, if E is a POVM on X covariant with respect to U, then there exists a density matrix invariant under rotations such that $E = E_T$.

The proof of the above theorem (which is a special case of a more general result [10]) is given in the next section, and it is based on the fact that G acts transitively on X, i.e. given any $x, y \in X$ it is always possible to find $g \in G$ such that $\alpha_g(x) = y$. In particular, the stability subgroup at the origin $(\vec{0}, \vec{0})$, i.e. the subgroup of elements of G acting trivially on the origin, is the compact group $SO(3)$, so that X is isomorphic to the quotient space $G/SO(3)$. The essential property involved in the proof of theorem 3 is the fact that U is *square-integrable* (see the definition in the next section). We will prove that square-integrability is a necessary and sufficient condition for the existence of covariant POVMs, when the stabilizer is compact.

Equation (11) can obviously also be written in terms of the Weyl operators according to

$$E_T(Z) = \frac{1}{(2\pi)^3} \int_Z \mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{Q} - \vec{a}\cdot\vec{P})} T \, \mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{Q} - \vec{a}\cdot\vec{P})} \, \mathrm{d}\vec{a} \, \mathrm{d}\vec{p}$$

where $\vec{Q}$ and $\vec{P}$ denote position and momentum operators acting on $L^2(\mathbb{R}^3, \mathrm{d}\vec{x})$.

We now characterize the positive trace class trace one operators T satisfying equation (10). We have the factorization

$$L^2(\mathbb{R}^3, \mathrm{d}\vec{x}) = L^2(S^2, \mathrm{d}\Omega) \otimes L^2(\mathbb{R}_+, r^2 \, \mathrm{d}r).$$

Denoting with l the representation of $SO(3)$ acting on $L^2(S^2, \mathrm{d}\Omega)$ by left translations, we have

$$U|_{SO(3)} = l \otimes I.$$

The representation $(l, L^2(S^2, \mathrm{d}\Omega))$ decomposes into

$$L^2(S^2, \mathrm{d}\Omega) = \bigoplus_{\ell \geqslant 0} M_\ell$$

where each irreducible inequivalent subspace M_ℓ is generated by the spherical harmonics $(Y_{\ell m})_{-\ell \leqslant m \leqslant \ell}$. We have

$$L^2(\mathbb{R}^3, \mathrm{d}\vec{x}) = \left(\bigoplus_{\ell \geqslant 0} M_\ell \right) \otimes L^2(\mathbb{R}_+, r^2 \, \mathrm{d}r) = \bigoplus_{\ell \geqslant 0} (M_\ell \otimes L^2(\mathbb{R}_+, r^2 \, \mathrm{d}r)).$$

Let $P_\ell : L^2(S^2, \mathrm{d}\Omega) \longrightarrow L^2(S^2, \mathrm{d}\Omega)$ be the orthogonal projection onto the subspace M_ℓ. If T intertwines $l \otimes I$, one has

$$T(P_\ell \otimes I) = (P_\ell \otimes I)T$$

where $P_\ell \otimes I$ projects onto $M_\ell \otimes L^2(\mathbb{R}_+, r^2 \, \mathrm{d}r)$. Given Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and an irreducible representation $(\pi, \mathcal{K})$, a standard result asserts that $\mathcal{C}(\pi \otimes I_{\mathcal{H}_1}, \pi \otimes I_{\mathcal{H}_2}) = I_{\mathcal{K}} \otimes \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Since M_ℓ is irreducible, this implies

$$T(P_\ell \otimes I) = P_\ell \otimes T_\ell$$

with $T_\ell \in \mathcal{L}(L^2(\mathbb{R}_+, r^2 \, \mathrm{d}r))$. We then have

$$T = \sum_\ell T(P_\ell \otimes I) = \sum_\ell P_\ell \otimes T_\ell.$$

In the last expression, T is a positive trace one operator if and only if each T_ℓ is positive and

$$1 \equiv \sum_\ell \dim M_\ell \operatorname{Tr} T_\ell = \sum_\ell (2\ell + 1) \operatorname{Tr} T_\ell. \tag{12}$$

It follows that the operators T associated with the U-covariant POVMs M by equation (11) are all the operators of the form

$$T = \sum_\ell P_\ell \otimes T_\ell \tag{13}$$

with T_ℓ positive trace class operators satisfying equation (12).

4. Proof of theorem 3

We prove theorem 3 in two steps. First, given an arbitrary topological group G and a compact subgroup H, we characterize all the POVMs based on the quotient space G/H and covariant with respect to an irreducible (ordinary) representation of G. Then, we apply the above result to our problem lifting the projective unitary representation U of the Galilei group to an (ordinary) unitary representation of the central extension G_ω of the Galilei group defined by the multiplier ω of U.

From now on, let G be a unimodular locally compact second countable topological group and H be a compact subgroup of G. We denote by

$$G \ni g \longmapsto \pi(g) = \dot{g} \in G/H$$

the canonical projection onto the quotient space G/H. Let μ_G and μ_H be invariant measures on G and H, respectively, with $\mu_H(H) = 1$. Due to the compactness of H, there exists a G-invariant measure $\mu_{G/H}$ on G/H such that the following measure decomposition holds:

$$\int_G f(g)\,\mathrm{d}\mu_G(g) = \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \int_H f(gh)\,\mathrm{d}\mu_H(h) \tag{14}$$

for all $f \in L^1(G, \mu_G)$.

Let U be an irreducible unitary representation U of G acting on the Hilbert space $\mathcal{H}$. We recall that U is said to be *square-integrable* if there exists a nonzero vector $\phi \in \mathcal{H}$ such that

$$\int_G |\langle\phi, U_g\phi\rangle_{\mathcal{H}}|^2\,\mathrm{d}\mu_G(g) < +\infty.$$

If the above condition holds, there exists a constant $d_U > 0$, called *formal degree*, such that for all $\phi \in \mathcal{H}$

$$\int_G |\langle\phi, U_g\phi\rangle_{\mathcal{H}}|^2\,\mathrm{d}\mu_G(g) = \frac{1}{d_U}\|\phi\|^4.$$

Finally, all the integrals of operator-valued functions (as, for example, in equation (15) below) are understood in the weak sense.

We need the following result which is proved in [10]. □

Proposition 4. *Assume that U is square-integrable with formal degree d_U and let T be positive trace class trace one operator $T \in \mathcal{B}(\mathcal{H})$. The map*

$$\mathcal{B}(G) \ni \widetilde{Z} \mapsto \widetilde{E}_T(\widetilde{Z}) = d_U \int_{\widetilde{Z}} U_g T U_g^*\,\mathrm{d}\mu_G(g) \tag{15}$$

defines a POVM $\widetilde{E}_T$ on G covariant with respect to U.

Conversely, if $\widetilde{E}$ is a POVM on G covariant with respect to U, then U is square-integrable and there is a trace class positive operator $T \in \mathcal{B}(\mathcal{H})$ with trace one such that $\widetilde{E} = \widetilde{E}_T$.

Now we extend the above result to covariant POVMs based on G/H.

Corollary 5. *Assume that U is a square-integrable representation with formal degree d_U and let T be a trace class positive operator $T \in \mathcal{B}(\mathcal{H})$ with trace one such that*

$$TU_h = U_h T \qquad \forall\, h \in H. \tag{16}$$

Then the map

$$\mathcal{B}(G/H) \ni Z \mapsto E_T(Z) = d_U \int_Z U_g T U_g^*\,\mathrm{d}\mu_{G/H}(\dot{g}) \tag{17}$$

defines a POVM E_T on G/H covariant with respect to U.

Conversely, if E is a POVM on G/H covariant with respect to U, then U is square-integrable and there is a trace class positive operator $T \in \mathcal{B}(\mathcal{H})$ with trace one and commuting with $U|_H$ such that $E = E_T$.

Proof. Assume that U is square-integrable and let $T \in \mathcal{B}(\mathcal{H})$ as in the statement of the corollary. By means of equation (15) T defines a POVM $\widetilde{E}_T$ based on G and covariant with respect to U. For all $Z \in \mathcal{B}(G/H)$ let

$$E_T(Z) = \widetilde{E}_T(\pi^{-1}(Z)).$$

Clearly, E_T is a POVM on G/H covariant with respect to U. Moreover, denoting with χ_Z the characteristic function of Z,

$$\begin{aligned} E_T(Z) &= d_U \int_G \chi_Z(\pi(g)) U_g T U_g^* \, \mathrm{d}\mu_G(g) \\ \text{(equation (14))} &= d_U \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \int_H \chi_Z(\pi(gh)) U_{gh} T U_{gh}^* \, \mathrm{d}\mu_H(h) \\ \text{(equation (16))} &= d_U \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \chi_Z(\dot{g}) U_g T U_g^* \end{aligned}$$

that is equation (17) holds.

Conversely, let E be a POVM on G/H and covariant with respect to U. For all $\widetilde{Z} \in \mathcal{B}(G)$, let $l_{\widetilde{Z}}$ be the function on G given by

$$l_{\widetilde{Z}}(g) = \mu_H(g^{-1}\widetilde{Z} \cap H) = \int_H \chi_{\widetilde{Z}}(gh) \, \mathrm{d}\mu_H(h).$$

Clearly, $l_{\widetilde{Z}}$ is a positive measurable function bounded by 1 and, since μ_H is invariant, for all $h \in H$, $l_{\widetilde{Z}}(gh) = l_{\widetilde{Z}}(g)$. It follows that there is a positive measurable bounded function $\ell_{\widetilde{Z}}$ on G/H such that $l_{\widetilde{Z}} = \ell_{\widetilde{Z}} \circ \pi$.

Define the operator $\widetilde{E}(\widetilde{Z})$ by means of

$$\widetilde{E}(\widetilde{Z}) = \int_{G/H} \ell_{\widetilde{Z}}(\dot{g}) \, \mathrm{d}E(\dot{g})$$

which is well defined since $\ell_{\widetilde{Z}}$ is bounded.

We claim that $\widetilde{Z} \mapsto \widetilde{E}(\widetilde{Z})$ is a POVM on G covariant with respect to U. Clearly, since $\ell_{\widetilde{Z}}$ is positive, $\widetilde{E}(\widetilde{Z})$ is a positive operator. Recalling that $\ell_G = 1$, one has $\widetilde{E}(G) = I$. Now let $(\widetilde{Z}_i)_i$ a disjoint sequence of $\mathcal{B}(G)$ and $\widetilde{Z} = \cup_i \widetilde{Z}_i$. Given $g \in G$, since $(g^{-1}\widetilde{Z}_i \cap H)_i$ is a disjoint sequence of $\mathcal{B}(H)$ and $g^{-1}\widetilde{Z} \cap H = \cup_i (g^{-1}\widetilde{Z}_i \cap H)$, then $\ell_{\widetilde{Z}} = \sum_i \ell_{\widetilde{Z}_i}$, where the series converges pointwise. Let $\phi \in \mathcal{H}$, by monotone convergence theorem, one has that

$$\langle \phi, \widetilde{E}(\widetilde{Z})\phi \rangle = \sum_i \langle \phi, \widetilde{E}(\widetilde{Z}_i)\phi \rangle.$$

Finally, let $g_1 \in G$, then

$$\begin{aligned} \widetilde{E}(g_1\widetilde{Z}) &= \int_{G/H} \mu_H(g^{-1}g_1\widetilde{Z} \cap H) \, \mathrm{d}E(\dot{g}) \\ (\dot{g} \mapsto g_1\dot{g}) &= \int_{G/H} \mu_H(g^{-1}\widetilde{Z} \cap H) U_{g_1} \, \mathrm{d}E(\dot{g}) U_{g_1}^* \\ &= U_{g_1} \widetilde{E}(\widetilde{Z}) U_{g_1}^* \end{aligned}$$

where we used the fact that E is covariant.

By means of proposition 4, U is square-integrable and there is a positive trace class operator one trace T such that

$$\widetilde{E}(\widetilde{Z}) = d_U \int_{\widetilde{Z}} U_g T U_g^* \, \mathrm{d}\mu_G(g). \tag{18}$$

We now show that T satisfies equation (16). First of all we claim that, given $h \in H$ and $\widetilde{Z} \in \mathcal{B}(G)$,

$$\widetilde{E}(\widetilde{Z}h) = \widetilde{E}(\widetilde{Z}). \tag{19}$$

Indeed, since H is compact, μ_H is both left and right invariant, so that

$$\mu_H(g^{-1}\widetilde{Z}h \cap H) = \mu_H((g^{-1}\widetilde{Z} \cap H)h) = \mu_H(g^{-1}\widetilde{Z} \cap H)$$

and, hence, $\ell_{\widetilde{Z}} = \ell_{\widetilde{Z}h}$. By definition of $\widetilde{E}(\widetilde{Z})$, equation (19) easily follows. Fixed $h \in H$, by means of equations (18) and (19), one has that

$$\int_{\widetilde{Z}} U_g T U_g^* \, \mathrm{d}\mu_G(g) = \int_{\widetilde{Z}h} U_g T U_g^* \, \mathrm{d}\mu_G(g) \qquad (g \mapsto gh) = \int_{\widetilde{Z}} U_{gh} T U_{gh}^* \, \mathrm{d}\mu_G(g)$$

where we used the fact that G is unimodular. Since the equality holds for all $\widetilde{Z} \in \mathcal{B}(G)$, then, for μ_G-almost all $g \in G$,

$$U_g T U_g^* = U_g U_h T U_h^* U_g^*$$

where the equality holds in the weak sense. Since both sides are continuous functions of g, the equality holds everywhere and equation (16) follows.

Let now $Z \in \mathcal{B}(G/H)$. Since

$$g^{-1}\pi^{-1}(Z) \cap H = \begin{cases} H & \text{if } gH \in Z \\ \emptyset & \text{if } gH \notin Z \end{cases}$$

then $\ell_{\pi^{-1}(Z)} = \chi_Z$ and $\widetilde{E}(\pi^{-1}(Z)) = E(Z)$. Reasoning as in the first part of the proof one has that $E = E_T$. □

Now we come back to the Galilei group G and to the projective unitary representation U of G associated with a spinless particle of mass m. We recall that *projective* means that for all $g_1, g_2 \in G$

$$U_{g_1} U_{g_2} = \omega(g_1, g_2) U_{g_1 g_2}$$

where ω is the *multiplier* given by

$$\omega((\vec{a}, \vec{v}, R), (\vec{a}', \vec{v}', R')) = \mathrm{e}^{\mathrm{i}m\vec{v}\cdot R\vec{a}'}.$$

We extend U to a unitary representation of the central extension G_ω of G associated with the multiplier ω (see, for example, [11]). Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the multiplicative group of the torus. The group G_ω is the product $\mathbb{T} \times G$ with the composition law

$$(z, \vec{a}, \vec{v}, R)(z', \vec{a}', \vec{v}', R') = (zz'\,\mathrm{e}^{\mathrm{i}m\vec{v}\cdot R\vec{a}'}, \vec{a} + R\vec{a}', \vec{v} + R\vec{v}', RR').$$

In particular, G_ω acts transitively on X by means of

$$\widetilde{\alpha}_{(z,\vec{a},\vec{v},R)}(\vec{q}, \vec{p}) = (\vec{a} + R\vec{q}, m\vec{v} + R\vec{p}) \tag{20}$$

and the stability subgroup at the origin is the compact subgroup $H = \mathbb{T} \times SO(3)$. In particular, X is isomorphic to G_ω/H by means of

$$(\vec{q}, \vec{p}) \mapsto \pi\left(1, \vec{q}, \frac{\vec{p}}{m}, I\right) \tag{21}$$

where $\pi : G_\omega \longrightarrow G_\omega/H$ is the canonical projection.

The irreducible projective representation U lifts to an irreducible unitary representation $\widetilde{U}$ of G_ω as

$$[\widetilde{U}_{(z,\vec{a},\vec{v},R)}\phi](\vec{x}) = z^{-1}\,\mathrm{e}^{\mathrm{i}m\vec{v}\cdot(\vec{x}-\vec{a})}\phi(R^{-1}(\vec{x}-\vec{a}))$$

where $\phi \in L^2(\mathbb{R}^3, \mathrm{d}\vec{x})$.

Clearly a POVM E is covariant with respect to U if and only if E is covariant with respect to $\widetilde{U}$. The classification of such POVMs is given in corollary 5. We only have to check that the representation $\widetilde{U}$ is square-integrable (compare with [12]). Indeed, an invariant measure of G_ω is

$$\mathrm{d}\mu_{G_\omega}(z,\vec{a},\vec{v},R) = \frac{m}{(2\pi)^3}\,\mathrm{d}z\,\mathrm{d}\vec{a}\,\mathrm{d}\vec{v}\,\mathrm{d}R$$

where $\mathrm{d}z$ and $\mathrm{d}R$ are normalized Haar measures in $\mathbb{T}$ and in $SO(3)$, respectively. Moreover, if $\phi \in L^2(\mathbb{R}^3, \mathrm{d}\vec{x})$, we have

$$\begin{aligned}
&\int_{G_\omega} |\langle\phi, \widetilde{U}_{(z,\vec{a},\vec{v},R)}\phi\rangle|^2\,\mathrm{d}\mu_{G_\omega}(z,\vec{a},\vec{v},R)\\
&\qquad= \int_{\mathbb{R}^3\times\mathbb{P}^3\times SO(3)\times\mathbb{T}} |\bar{z}\langle\phi, \widetilde{U}_{(1,\vec{a},\vec{v},R)}\phi\rangle|^2\frac{m\,\mathrm{d}\vec{a}\,\mathrm{d}\vec{v}\,\mathrm{d}R\,\mathrm{d}z}{(2\pi)^3}\\
&\qquad= \int_{\mathbb{R}^3\times\mathbb{P}^3\times SO(3)} \left|\int_{\mathbb{R}^3}\phi(\vec{x})\,\mathrm{e}^{-\mathrm{i}m\vec{v}\cdot(\vec{x}-\vec{a})}\overline{\phi(R^{-1}(\vec{x}-\vec{a}))}\,\mathrm{d}\vec{x}\right|^2\frac{m\,\mathrm{d}\vec{a}\,\mathrm{d}\vec{v}\,\mathrm{d}R}{(2\pi)^3}\\
&\qquad= \int_{\mathbb{R}^3\times SO(3)}\left[\int_{\mathbb{P}^3}|\mathcal{F}(\phi(\cdot)\overline{\phi(R^{-1}(\cdot-\vec{a}))})(m\vec{v})|^2 m\,\mathrm{d}\vec{v}\right]\mathrm{d}\vec{a}\,\mathrm{d}R\\
&\qquad= \int_{\mathbb{R}^3\times SO(3)}\left[\int_{\mathbb{R}^3}|\phi(\vec{x})\overline{\phi(R^{-1}(\vec{x}-\vec{a}))}|^2\,\mathrm{d}\vec{x}\right]\mathrm{d}\vec{a}\,\mathrm{d}R = \|\phi\|^4.
\end{aligned}$$

Here $\mathcal{F}$ denotes the Fourier transform.

Then, choosing $\mathrm{d}\mu_{G_\omega/H}(\vec{a},\vec{v}) = \frac{m}{(2\pi)^3}\,\mathrm{d}\vec{a}\,\mathrm{d}\vec{v}$, one has $d_{\widetilde{U}} = 1$, and every $\widetilde{U}$-covariant POVM based on G_ω/H has the form

$$E_T(Z) = \frac{m}{(2\pi)^3}\int_Z \widetilde{U}_{(1,\vec{a},\vec{v},I)}T\widetilde{U}^*_{(1,\vec{a},\vec{v},I)}\,\mathrm{d}\vec{a}\,\mathrm{d}\vec{v} \tag{22}$$

for all $Z \in \mathcal{B}(G_\omega/H)$, where T is a positive trace one operator commuting with $\widetilde{U}|_{\mathbb{T}\times SO(3)}$. Clearly, T commutes with $\widetilde{U}|_{\mathbb{T}\times SO(3)}$ if and only if it commutes with $U|_{SO(3)}$. Taking into account the identification between X and G_ω/H given by equation (21), the proof of theorem 3 is complete.

Remark 6. One can prove that the representation $\widetilde{U}$ is square-integrable by an abstract argument. Indeed, G_ω is the semidirect product of the normal Abelian closed subgroup $\mathbb{T}\times\mathbb{R}^3$ and the closed subgroup $\mathbb{V}^3 \times' SO(3)$. Moreover, $\widetilde{U}$ is the representation unitarily induced by σ from $\mathbb{T}\times\mathbb{R}^3\times SO(3)$ to G_ω, where σ is the representation of $\mathbb{T}\times\mathbb{R}^3\times SO(3)$ acting on $\mathbb{C}$ as

$$\sigma_{(z,\vec{x},R)} = z^{-1}.$$

The corresponding orbit in the dual group $\widehat{\mathbb{T}\times\mathbb{R}^3} = \mathbb{Z}\times\mathbb{P}^3$ is $\mathcal{O} = \{-1\}\times\mathbb{P}^3$. Since $\mathcal{O}$ has a strictly positive measure (with respect to the Haar measure of $\mathbb{Z}\times\mathbb{P}^3$) and $\sigma|_{SO(3)}$ is square-integrable, a theorem proved in [13] assures that $\widetilde{U}$ is square-integrable.

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